SOME PROPERTIES OF BETA TRANSMUTED DAGUM DISTRIBUTION WITH APPLICATIONS*

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Abstract

In this paper, we introduce a new family of continuous distributions called the beta transmuted Dagum distribution which extends the beta and transmuted familys. The genesis of the beta distribution and transmuted map is used to develop the so-called beta transmuted Dagum (BTD) distribution. The hazard function, moments, moment generating function, quantiles and stress-strength of the beta transmuted Dagum distribution (BTD) are provided and discussed in detail. The method of maximum likelihood estimation is used for estimating the model parameters. A simulation study is carried out to show the performance of the maximum likelihood estimate of parameters of the new distribution. The usefulness of the new model is illustrated through an application to a real data set.

Keywords: beta dagum distribution, dagum distribution, maximum likelihood method, moments, transmuted distribution.

1. Introduction

Dagum proposed a three-parameter (type I) and four-parameter (type II) distributions for modeling size distribution of income in Dagum (1980, 2008). However, the Dagum type I (or Dagum) distribution has received increased attention just because of being a tentative competitor as compared to other models. A detailed discussion on the Dagum distributions is addressed in Dagum (1997, 2006). In fact, the Dagum model has many

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properties that are required for describing an income size model. The importance of this model is that it provides good fit to the extreme sides of income data. Its applications to human capital and personal income appeared in Costa (2006), Pérez & Alaiz (2011), Ivana (2011), and Lukasiewicz et al. (2012). Binoti et al. (2012) and Alwan et al. (2013) worked with the Dagum distribution for assessing the reliability of an electrical system and for describing diameter in teak stands subjected to thinning at different ages. Kleiber & Kotz (2003), Shahzad & Asghar (2013), and Pant & Headrick (2013) discussed properties and parameter estimation of the Dagum distribution. Domma (2007) determined the asymptotic distribution of the maximum likelihood estimators (MLEs) for the right-truncated Dagum model. Pollastri & Zambruno (2010) proposed an estimation procedure of the distribution of the ratio of two independent Dagum random variables. Domma et al. (2012) described the usefulness of the Dagum model in reliability theory and showed that its hazard rate function (hrf) can have a decreasing, an upside-down bathtub and a bathtub and then upside-down bathtub forms. Domma et al. (2011) discussed the maximum likelihood estimation of the Dagum's parameters for censored data which usually occur in life-testing problems. The Fisher information matrix of doubly censored data and type II doubly censored data was computed in Domma et al. (2013). Domma (2004) defined the log-Dagum distribution and studied the changes in the kurtosis by using the kurtosis diagram given by Zenga (1996) and Polisicchio & Zenga (1997). Domma & Perri (2009) discussed some more structural properties and parameter estimation of the log-Dagum distribution. A random variable X has the Dagum (Type-I) distribution with three positive parameters α, θ and β, if its cdf is given by

$$G(x) = \left(1 + \alpha x^{-\theta}\right)^{-\beta}, x > 0, \tag{1}$$

where α is a scale parameter, and θ and β are shape parameters, the pdf corresponding to (1) is given by

$$g(x) = \alpha\theta\beta x^{-\theta-1}\left(1 + \alpha x^{-\theta}\right)^{-\beta-1}, \text{ for } \alpha, \theta, \beta > 0 \tag{2}$$

Recently, Elbatal & Aryal (2015) introduced another generalization of the Dagum distribution, which they called the transmuted Dagum distribution. A random variable X is said to have transmuted Dagum probability distribution with parameters $\alpha > 0, \theta > 0,\ \beta > 0\ and\ |\lambda| \leq 1$, the cdf of transmuted Dagum distribution

$$\mathrm{G(x)} = \left(1 + \alpha x^{-\theta}\right)^{-\beta}\ \left(1 + \lambda - \lambda\left(1 + \alpha x^{-\theta}\right)^{-\beta}\right), \tag{3}$$

The probability density function (pdf) of the transmuted Dagum distribution is given by

$$\mathrm{g(x)} = \alpha\theta\beta x^{-\theta-1}\left(1 + \alpha x^{-\theta}\right)^{-\beta-1}\ \left(1 + \lambda - 2\lambda\left(1 + \alpha x^{-\theta}\right)^{-\beta}\right). \tag{4}$$

A class of generalized distributions $\mathrm{F(x)}$ has been receiving considerable attention over the last few years, in particular, after the studies by Eugene et al. (2002) and Jones (2004). If G denotes the baseline cumulative distribution function (cdf) of a random variable, then the beta-G distribution is defined as

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt, \tag{5}$$

where $a > 0$ and $b > 0$ are shape parameters. Note that $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$, is the incomplete beta function ratio, and $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}\ dt$, is the incomplete beta function, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function and $\Gamma(.)$ is the gamma function. The probability density function (pdf) of the Beta-G distribution has the form

$$\mathrm{f(x)} = \frac{\mathrm{g(x)}}{\mathrm{B}(a,\mathrm{b})}\ [\mathrm{G}(x)]^{\mathrm{a}-1}[1-\mathrm{G}(x)]^{\mathrm{b}-1}. \tag{6}$$

This class of generalized distribution has received considerable attention over the last years and several classical distributions have been generalized using this formulation. We generalize the transmuted Dagum distribution (3) using this formulation in order to construct the beta transmuted Dagum (BTD) distribution. This paper is organized as follows. In section 2, we define the BTD distribution and discuss some of its sub-models. In Section 3 we present the mixture representation of the BTD distribution. Section 4 discusses some structural and mathematical properties of the BTD distribution such as the moments, quantile and stress-strength reliability. The maximum likelihood estimation of the model parameters and a simulation study are investigated in Section 5. Applications are given in Section 6, followed by concluding remarks.

2. **The Beta Transmuted Dagum Distribution**

We provide the formulation of the beta transmuted Dagum (BTD) distribution. By inserting (3) into (5) the cumulative distribution function of the beta-transmuted Dagum distribution with four parameters is given by

$$F(x) = I_{(1+\alpha x^{-\theta})^{-\beta}\ \left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)}(a,b),$$

$$= \frac{1}{B(a,b)}\int_0^{(1+\alpha x^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)} t^{a-1}(1-t)^{b-1}dt, \tag{7}$$

where $x > 0, \alpha > 0$, $\theta > 0$, $\beta > 0, |\lambda| \le 1$ and $a > 0, b > 0$.

The cdf can be expressed in a closed form using the hypergeometric function see Cordeiro & Nadarajah (2011) as follows:

$$F(x) = \frac{\left[(1+\alpha x^{-\theta})^{-\beta}\ \left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)\right]^a}{aB(a,b)} . 2F1\Big(a, 1-b; a + 1; (1+\alpha x^{-\theta})^{-\beta}\ \left(1+\lambda-\lambda(1+\alpha x^{-\theta})^{-\beta}\right)\Big),$$

where

$$2F1(c,d;e;z) = \sum_{k=0}^{\infty} \frac{(c)_k\ (d)_k}{(e)_k}\ \frac{(z^k)}{k!}$$

is the Gaussian hypergeometric function with $(c)_k$ defined as

$$(c)_k = \begin{cases} c(c+1)(c+2)\ldots.(c+k-1) & k = 1,2,3,\ldots \\ 1 & k = 0. \end{cases}$$

The pdf $f(x)$ and the hazard rate function $h(x)$ are obtained as

$$f(\mathrm{x}) = \frac{\alpha\theta\beta}{\mathrm{x}^{\theta+1}\,\mathrm{B(a,b)}}(1+\alpha \mathrm{x}^{-\theta})^{-\beta-1}\left(1+\lambda-2\lambda(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right)\left[(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right]^{\mathrm{a}-1}\left[\left(1+\lambda-\lambda(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right)\right]^{\mathrm{a}-1}\left[1-(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right)\right]^{\mathrm{b}-1}, \tag{8}$$

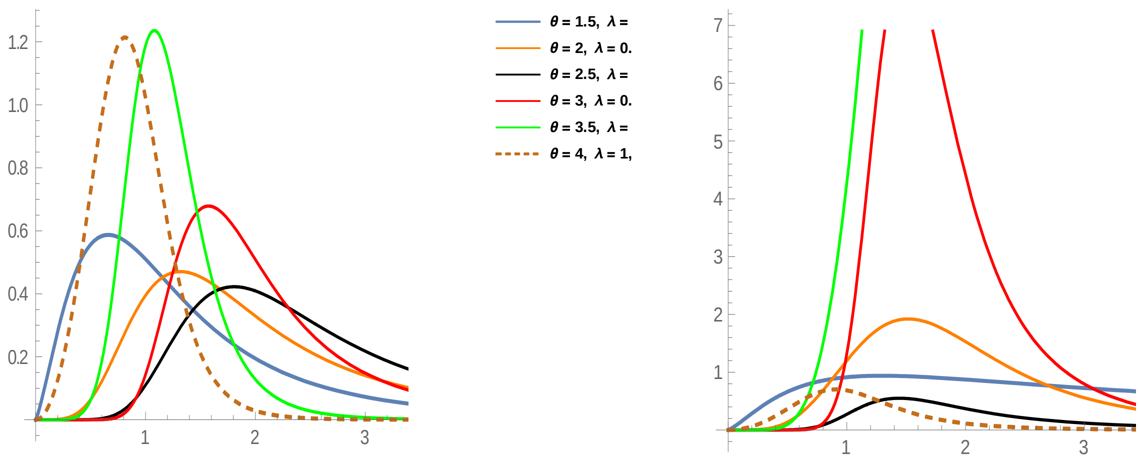
$$\mathrm{h(x)} = \frac{\alpha\theta\beta\,\mathrm{x}^{-\theta-1}\,(1+\alpha \mathrm{x}^{-\theta})^{-\beta-1}\left(1+\lambda-2\lambda(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right)}{\mathrm{B(a,b)}\,I_{1-\left(1+\alpha \mathrm{x}^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha \mathrm{x}^{-\theta}\right)^{-\beta}\right)}(\mathrm{a,b})}\left[(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right]^{\mathrm{a}-1}\left[\left(1+\lambda-\lambda(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right)\right]^{\mathrm{a}-1}\left[1-(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha \mathrm{x}^{-\theta})^{-\beta}\right)\right]^{\mathrm{b}-1}, \tag{9}$$

where $x > 0, \alpha > 0,\ \theta > 0\ , \beta > 0, |\lambda| \leq 1$ and $a > 0, b > 0$.

The BTD distribution includes the following distributions as special case:

- for $\lambda = 0$, beta transmuted Dagum reduces to beta Dagum distribution.
- For $\mathrm{a} = \mathrm{b} = 1$, beta transmuted Dagum reduces to transmuted Dagum distribution.
- For $\mathrm{a} = \mathrm{b} = 1$ and $\lambda = 0$, beta transmuted Dagum reduces to Dagum distribution.

Figure 1 illustrates the graphical behavior of the pdf (8) and the hazard rate function (9) of BTD distribution for selected values of the parameters θ, λ and a with $\alpha = 1.5,\ \beta = 1.5\ and\ b = 1$.



(a) (b)

Figure 1: Pdf (a) and $h(x)$ (b) of BTD distribution for selected values of the parameters.

3. Mixture Representation

In this section we find the series representations of the cdf and the pdf of the BTD distribution which will be useful to study its mathematical characteristics. For an arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G ("exp-G" for short) distribution with power parameter $a > 0$, say $Y \sim exp - G(a)$, if its cdf and pdf are $H_a(x) = [G(x)]^a$ and $h_a(x) = ag(x)[G(x)]^{a-1}$, respectively. As we shall see both pdf and cdf of BTD distribution can be expressed in terms of the Dagum distribution. Consider the power series expansion

$$(1-t)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} [G(x)]^j,$$

which holds for $|t| < 1$ and $b > 0$ real non-integer. The pdf in (8) can be rewritten as

$$\mathrm{f(x)} = \frac{\mathrm{g(x)}}{\mathrm{B(a,b)}} \left(1+\lambda-2\lambda G(x)\right) \quad [(1+\lambda)G(x) - \lambda G(x)^2)]^{\mathrm{a}-1}[1-\{((1+\lambda)G(x) - \lambda G(x)^2)\}]^{\mathrm{b}-1}.$$

Consider $A = [1-\{((1+\lambda)G(x) - \lambda G(x)^2)\}]^{\mathrm{b}-1}$.

Applying the power series to the quantity A, we obtain

$$\mathrm{f(x)} = \frac{1}{\mathrm{B(a,b)}} \underbrace{\mathrm{g(x)} \left(1+\lambda-2\lambda G(x)\right)}_{h(x)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k)}{k!\ \Gamma(b-k)} \underbrace{[(1+\lambda)G(x) - \lambda G(x)^2]^{a+k-1}}_{H(x)^{a+k-1}}$$

Further, we can write the last equation as

$$\mathrm{f(x)} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k)}{\mathrm{B(a,b)}k!\ \Gamma(b-k)} \mathrm{g(x)}\ G(x)^{a+k-1}.$$

Finally, the above pdf can be expressed as a mixture of exp-G pdfs

$$f(x) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl}\ h_{k+l}(x), \tag{10}$$

where $h_{k+l}(x)$ is the exponentiated Dagum (ED) density with parameters α, θ, β and $k+l$ given by

$$h_{k+l}(x) = (k+l)\alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-(k+l)\beta-1}$$

and $w_k = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k)}{\mathrm{B(a,b)}k!\ \Gamma(b-k)(a+k)}$.

Thus, several mathematical properties of the BTD can be obtained simply from those properties of the exp-G family. Equation (10) is the main result of this section. The cdf of the BTG can also be expressed as a mixture of exp-G cdfs. By integrating (10), we obtain the mixture representation

$$F(x) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl}\ H_{k+l}(x), \tag{11}$$

where $H_{k+l}(x) = G(x)^{k+l}$ is the cdf of the exponentiated Dagum (ED) with parameters α, θ, β and power parameter $k + l$.

4. Mathematical Characterizations

In this section we provide some mathematical properties of the BTD distribution including the moments and moment generating function, quantiles and stress-strength model.

4.1 Moments and moments generating function

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). If X has the $BTD(\alpha, \theta, \beta, \lambda, a, b)$ then the r^{th} moment of X are given by the following

$$E(X^r) = \int_0^\infty x^r f(x)dx = \alpha^{\frac{r}{\theta}} B\left(\beta + \frac{r}{\theta},\ 1 - \frac{r}{\theta}\right) \sum_{k,l=0}^{\infty} w_{kl}\ (k+l)^{\frac{r}{\theta}}. \qquad (12)$$

The r^{th} moment will be defined only when $\theta > r$. In particular,

$$E(X) = \alpha^{\frac{1}{\theta}} B\left(\beta + \frac{1}{\theta}, 1 - \frac{1}{\theta}\right) \sum_{k,l=0}^{\infty} w_{kl}\ (k+l)^{\frac{1}{\theta}}, \quad \text{if } \theta > 1,$$

The variance, skewness and kurtosis of the BTD distribution can be calculated from (12) using the relations given below.

$$\mathrm{Var}(X) = \alpha^{\frac{2}{\theta}} \frac{\Gamma(\beta)\Gamma\left(\beta+\frac{2}{\theta}\right)\Gamma\left(\frac{-2+\theta}{\theta}\right) - \Gamma\left(\beta+\frac{1}{\theta}\right)^2 \Gamma\left(\frac{-1+\theta}{\theta}\right)^2}{\Gamma(\beta)^2} \sum_{k,l=0}^{\infty} w_{kl}\ (k+1)^{\frac{2}{\theta}},$$

$$\mathrm{Skewness}(X) = \sum_{k,l=0}^{\infty} w_{kl}\ (k+1)^{\frac{3}{\theta}} \alpha^{\frac{3}{\theta}} \left[\left(\Gamma(\beta)^2 \Gamma\left(\beta + \frac{3}{\theta}\right) \Gamma\left(\frac{-3+\theta}{\theta}\right) - 3\Gamma(\beta)\Gamma\left(\beta+\frac{1}{\theta}\right)\Gamma\left(\beta+\frac{2}{\theta}\right) \Gamma\left(\frac{-2+\theta}{\theta}\right)\Gamma\left(\frac{-1+\theta}{\theta}\right) + 2\,\Gamma\left(\beta+\frac{1}{\theta}\right)^2 \Gamma\left(\frac{-1+\theta}{\theta}\right)^3 \right) \Big/ \Gamma(\beta)^3 \left(\frac{\Gamma(\beta)\Gamma\left(\beta+\frac{2}{\theta}\right)\Gamma\left(\frac{-2+\theta}{\theta}\right) - \Gamma\left(\beta+\frac{1}{\theta}\right)^2 \Gamma\left(\frac{-1+\theta}{\theta}\right)^2}{\Gamma(\beta)^2} \right)^{\frac{3}{2}} \right],$$

$$\mathrm{Kurtosis}(X) = \sum_{k,l=0}^{\infty} w_{kl} \left[-3 \left(\Gamma(\beta)^2 \left(\Gamma(\beta)\Gamma\left(\beta+\frac{4}{\theta}\right) \Gamma\left(\frac{-4+\theta}{\theta}\right) + 3\Gamma\left(\beta+\frac{2}{\theta}\right)^2 \Gamma\left(\frac{-2+\theta}{\theta}\right)^2 - 4\Gamma\left(\beta+\frac{1}{\theta}\right)\Gamma\left(\beta+\frac{3}{\theta}\right) \Gamma\left(\frac{-3+\theta}{\theta}\right)\Gamma\left(\frac{-1+\theta}{\theta}\right) \right) \right) \Big/ \left(\Gamma(\beta)\Gamma\left(\beta+\frac{2}{\theta}\right) \Gamma\left(\frac{-2+\theta}{\theta}\right) - \Gamma\left(\beta+\frac{1}{\theta}\right)^2 \Gamma\left(\frac{-1+\theta}{\theta}\right)^2 \right)^2 \right].$$

The moment generating function, MGF, of a random variable X is defined by

$M_X(t) = E(e^{tx})$. When X has the BTD $(\alpha, \theta, \beta, \lambda, a, b)$ then the MGF of X is given by

$$M_X(t) = E(e^{tx}) = \sum_{k,l=0}^{\infty} w_{kl} \sum_{r=0}^{\infty} \frac{t^r}{r!}\, \beta\, \left(\alpha(k+l)\right)^{\frac{r}{\theta}} B\left(\beta + \frac{r}{\theta}, \quad 1 - \frac{r}{\theta}\right), \quad \theta > r. \tag{13}$$

4.2 Quantiles

Quantile functions are in widespread use in statistics and often find representations in terms of lookup tables for key percentiles. The quantile function of a distribution is the real solution of $F(x_q) = q$ for $0 \le q \le 1$. The quantiles of BTD distribution are obtained from (7) as

$$X = Q(u) = \alpha^{\frac{1}{\theta}} \left[-1 + \left(\frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - 4\lambda\left(I_u^{-1}(a,b)\right)}}{2\lambda} \right)^{\frac{-1}{\beta}} \right]^{\frac{-1}{\theta}} \tag{14}$$

where $I_u^{-1}(a, b)$ is the inverse of the incomplete beta function with parameters a and b. As shown in Zea et al. (2012).

4.3 Stress-strength model

A stress-strength model describes the life of a component which has a random strength X_1 and is subjected to a random stress X_2. The component functions satisfactorily as long as $X_1 > X_2$, and fails when $X_1 < X_2$. The probability $R = Pr(X_1 > X_2)$ defines the component reliability. Stress-strength models have many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

Consider X_1 and X_2 to be independently distributed, with $X_1 \sim BTD(\alpha_1, \theta, \beta, \lambda_1, a_1, b_1)$ and $X_2 \sim BTR(\alpha_2, \theta, \beta, \lambda_2, a_2, b_2)$. The cdf $F_1$of X_1 and pdf f_2 of X_2 obtained from (10) and (11), respectively. Then,

$$R = Pr(X_1 > X_2) = \int_0^{\infty} f_2(y)[1 - F_1(y)]dy$$

$$= 1 + \sum_{k,l=0}^{\infty} w_{kl}^{(1)} \int_0^{\infty} f_2(y)(1 + \alpha(k+l)x^{-\theta})^{-\beta} dy = \sum_{k,l=0}^{\infty} w_{kl}^{(1)}\, A(k,l),$$

where

$$w_{kl}^{(i)} = \sum_{j=0}^{\infty} (-1)^{j+k+l} \binom{b_i - 1}{j} \binom{a_i + j}{k} \binom{a_i + j}{l} \frac{\lambda^l}{B(a,b)(a_i + j)} \qquad i = 1,2,$$

and

$$A(k,l) = \int_0^{\infty} f_2(y)(1+\alpha(k+l)x^{-\theta})^{-\beta} dy.$$

Now,

$$A(k,l) = \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \int_0^{\infty} (r+s)\theta\beta\, \alpha_2 x^{-\theta-1}\left[\left\{(1+\left(\alpha_2(r+s)+\alpha_1(k+l)\right)x^{-\theta})^{-\beta-1}\right\}\right]dy$$

$$= \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \frac{(r+s)\alpha_2}{(k+l)\alpha_1+(r+s)\alpha_2}.$$

Hence,

$$R = 1 + \sum_{k,l=0}^{\infty} w_{kl}^{(1)} \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \frac{(r+s)\alpha_2}{(k+l)\alpha_1+(r+s)\alpha_2},$$

$$= 1 + \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} w_k^{*(1)}\, w_r^{*(2)} \frac{r\alpha_2}{k\alpha_1 + r\alpha_2}, \tag{15}$$

where

$$\mathrm{w}_{\mathrm{m}}^{*(\mathrm{i})} = \sum_{\mathrm{k,l:k+l=m}} \mathrm{w}_{\mathrm{kl}}^{*(\mathrm{i})}, \quad \mathrm{i} = 1,2.$$

5. Parameter Estimation

In this section we will discuss about the method of parameter estimation of the beta transmuted Dagum distribution. The Maximum Likelihood Estimation is one of the most widely used estimation method for finding the unknown parameters. Asymptotic distribution of $\widehat{\Theta} = \left(\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{\lambda}, \hat{a}, \hat{b}\right)$ are obtained using the elements of the inverse Fisher information matrix.

5.1 Maximum Likelihood Estimation

Consider a random sample $x_1, x_2, \ldots, x_n$ from $X \sim BTD(\alpha,\theta,\beta,\lambda,a,b)$ distribution. The likelihood function can be written as

$$L(\Theta) = \left(\frac{\alpha\theta\beta}{\mathrm{B(a,b)}}\right)^n \prod_{i=1}^{n} \mathrm{x_i}^{-\theta-1} \left(1+\alpha\mathrm{x_i}^{-\theta}\right)^{-\beta-1} \left(1+\lambda-2\lambda(1+\alpha\mathrm{x_i}^{-\theta})^{-\beta}\right) \left[\left((1+\alpha\mathrm{x_i}^{-\theta})^{-\beta}\right)\right]^{\mathrm{a}-1} \left[\left(1+\lambda-\lambda(1+\alpha\mathrm{x_i}^{-\theta})^{-\beta}\right)\right]^{\mathrm{a}-1} \left[1-(1+\alpha\mathrm{x_i}^{-\theta})^{-\beta}\left(1+\lambda-\lambda(1+\alpha\mathrm{x_i}^{-\theta})^{-\beta}\right)\right]^{\mathrm{b}-1}.$$

Now, the log-likelihood function denoted by $l(\Theta)$:

$$l(\Theta) = n\log\alpha + n\log\theta + n\log\beta - n\log[B(a,b)] - (\theta+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n} Log\left[1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right] + (a-1)\left[\sum_{i=1}^{n} Log\left(\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right) + \sum_{i=1}^{n} Log\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right] + (b-1)\sum_{i=1}^{n} Log\left[1-\left(\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right], \quad (16)$$

The entries of the score function is given by

$$\frac{\partial l(\Theta)}{\partial\alpha} = \frac{n}{\alpha} - \beta(a-1)\sum_{i=1}^{n}\frac{x_i^{-\theta}}{1+\alpha x_i^{-\theta}} + 2\beta\lambda\sum_{i=1}^{n}\frac{x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + \beta\lambda(a-1)\sum_{i=1}^{n}\frac{x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} - \beta(b-1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-2\beta-1} + x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}, \quad (17)$$

$$\frac{\partial l(\Theta)}{\partial\theta} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log x_i + \alpha\beta(a-1)\sum_{i=1}^{n}\frac{\log x_i\ x_i^{-\theta}}{1+\alpha x_i^{-\theta}} - 2\beta\lambda\sum_{i=1}^{n}\frac{\log x_i\ x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} - \alpha\beta\lambda(a-1)\sum_{i=1}^{n}\frac{\log x_i\ x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}}{\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + \alpha\beta(b-1)\sum_{i=1}^{n}\frac{\lambda\log x_i\ x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-2\beta-1} + \log x_i\ \ x_i^{-\theta}\left(1+\alpha x_i^{-\theta}\right)^{-\beta-1}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}, \quad (18)$$

$$\frac{\partial l(\Theta)}{\partial\beta} = \frac{n}{\beta} - (a-1)\sum_{i=1}^{n}\log\left(1+\alpha x_i^{-\theta}\right) + 2\lambda\sum_{i=1}^{n}\frac{\log\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + \lambda(a-1)\sum_{i=1}^{n}\frac{\log\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + (b-1)\sum_{i=1}^{n}\frac{-\lambda\log\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-2\beta} + \log\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}, \quad (19)$$

$$\frac{\partial l(\Theta)}{\partial\lambda} = \sum_{i=1}^{n}\frac{1-2\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-2\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} + (a-1)\sum_{i=1}^{n}\frac{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}}{\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)} - (b-1)\sum_{i=1}^{n}\frac{\left(1+\alpha x_i^{-\theta}\right)\left(1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right) + \log\left(1+\alpha x_i^{-\theta}\right)\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}{1-\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)}, \quad (20)$$

$$\frac{\partial l(\Theta)}{\partial a} = -n[\Psi(a)-\Psi(a+b)] + \sum_{i=1}^{n} Log\left(1+\alpha x_i^{-\theta}\right)^{-\beta} + \sum_{i=1}^{n} Log\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right), \quad (21)$$

$$\frac{\partial l(\Theta)}{\partial b} = -n[\Psi(b)-\Psi(a+b)] + \sum_{i=1}^{n} Log\left[1-\left(\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\left(1+\lambda-\lambda\left(1+\alpha x_i^{-\theta}\right)^{-\beta}\right)\right], \quad (22)$$

Where $\Psi(x)$ is the digamma function defined by $\Psi(x) = \frac{d\log\Gamma(x)}{dx}$, and $\Gamma(x)$ is the Gamma function. The maximum likelihood estimators $\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{\lambda}, \hat{a}, \hat{b}$ of the unknown parameters $\alpha, \theta, \beta, \lambda, a, b$ respectively, can be obtained by setting the score vector to zero and solving the system of nonlinear equations simultaneously. Since there is no closed form solution of these non-linear system of equations, we can use numerical methods such as the quasi-Newton algorithm to numerically optimize the log-likelihood function given in (16) to get the maximum likelihood estimates of the parameters $\alpha, \theta, \beta, \lambda, a, b$. To study the properties of the estimators of the beta transmuted Dagum distribution and their performances. There are many measures that can be used to get information about the performance of the estimators. The bias and mean square error (MSE) are such useful measures.

5.2 Simulation Study

A simulation study is carried out to investigate the performance of the MLEs. We take sample sizes to be $n \in \{10,50,100,150,200,300,500\}$, and generate observations from a BTD distribution with parameters $\alpha = 0.9, \theta = 0.8, \beta = 0.7, \lambda = 0.6, a = 0.5$ and $b = 0.4$. The bias and MSE are calculated by:

$$\mathrm{Bias}(\widehat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\widehat{\Theta}_i - \Theta)$$

and

$$\mathrm{MSE}(\widehat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\widehat{\Theta}_i - \Theta)^2$$

respectively. The process is replicated 1000 times, and the average bias, along with the mean squared error are presented in Table 1.

Table1: Bias and MSE (in parentheses) for the BTD distribution.

n	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
10	0.745241	0.851783	0.497498	0.952656	0.698317	0.938319
	(0.57569)	(1.26836)	(2.24467)	(0.90786)	(0.49047)	(0.88701)
50	0.530543	0.640307	0.333568	0.724872	0.549683	0.659199
	(0.37818)	(1.03910)	(1.77811)	(0.81530)	(0.30219)	(0.57458)
100	0.346745	0.429959	0.304044	0.501654	0.416393	0.425698
	(0.21526)	(0.84866)	(1.55107)	(0.63214)	(0.23375)	(0.41286)
150	0.155984	0.242809	0.122452	0.348147	0.309345	0.316724
	(0.12387)	(0.53269)	(1.22993)	(0.40652)	(0.19098)	(0.30759)
200	0.109292	0.178652	0.102153	0.206416	0.138416	0.164331
	(0.02694)	(0.28369)	(1.10462)	(0.21292)	(0.10986)	(0.10547)
300	0.035116	0.030129	0.032129	0.080833	0.105385	0.066245
	(0.02091)	(0.12121)	(0.74534)	(0.06231)	(0.08965)	(0.04909)
500	0.014638	0.012033	0.010915	0.039743	0.038246	0.032586
	(0.01032)	(0.01631)	(0.44129)	(0.01888)	(0.02705)	(0.01825)

From Table 1 it is observed that as the sample size increases, the average biases and mean squared errors decrease. This verifies the consistency properties of the estimates.

6. Application

In this section, we provide application to the real data set to prove empirically the potentiality of the BTD distribution. We also compare the fits of the BTD, transmuted Dagum (TD), beta Dagum (BD) and Dagum distributions by means of data set. The data set corresponds to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee & Wang (2003). This data is given in Table 2 and The descriptive summary of the data is provided in Table 3.

Table 2: Data on 128 bladder cancer patients.

0.080	2.090	3.480	4.870	6.940	8.660	13.11	23.63	0.200	2.230
3.520	4.980	6.970	9.020	13.29	0.400	2.260	3.570	5.060	7.090
9.220	13.80	25.74	0.500	2.460	3.640	5.090	7.260	9.470	14.24
25.82	0.510	3.700	2.540	5.170	7.280	9.740	14.76	26.31	0.810
2.620	3.820	5.320	7.320	10.06	14.77	32.15	2.640	3.880	5.320
7.390	10.34	14.83	34.26	0.900	2.690	4.180	5.340	7.590	10.66
15.96	36.66	1.050	2.690	4.230	5.410	7.620	10.75	16.62	43.01
1.190	2.750	4.260	5.410	7.630	17.12	46.12	1.260	2.830	4.330
5.490	7.660	11.25	17.14	79.05	1.350	2.870	5.620	7.870	11.64
17.36	1.400	3.020	4.340	5.700	7.930	11.79	18.10	1.460	4.400
5.850	8.260	11.98	19.13	1.760	3.250	4.500	6.250	8.370	12.02
2.020	3.310	4.510	6.540	8.530	12.03	20.28	2.020	3.360	6.760
12.07	21.73	2.070	3.360	6.930	8.650	12.63	22.69		

To determine the optimum model, we also compute the estimated log-likelihood values $\hat{l}$, Akaike Information Criteria (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan Quinn information criterion (HQIC), Anderson-Darling (A^*), Crameér–von Mises (W^*) and Kolmogorov Smirnov (K-S) to compare the four fitted models. The statistics AIC, CAIC, BIC, HQIC, A^*, W^* and K-S are given by

$$AIC = 2K - 2l(\hat{\Theta}),$$

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = klog(n) - 2l(\hat{\Theta}),$$

$$HQIC = 2klog(log(n)) - 2l(\hat{\Theta}),$$

$$A^* = A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\left(F_{BTD}(x_i, \hat{\alpha}, \widehat{\theta}, \hat{\beta}, \hat{\lambda}, \hat{a}, \hat{b})\right) + log\left(1 - F_{BTD}(x_i, \hat{\alpha}, \widehat{\theta}, \hat{\beta}, \hat{\lambda}, \hat{a}, \hat{b})\right)\right]^2,$$

$$W^* = W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_{BTD}(x_i, \hat{\alpha}, \widehat{\theta}, \hat{\beta}, \hat{\lambda}, \hat{a}, \hat{b}) - \frac{2i-1}{2n}\right]^2,$$

and

$$K-S=\max_{i}\left[\frac{i}{n}-F_{BTD}\left(x_i,\hat{\alpha},\hat{\theta},\hat{\beta},\hat{\lambda},\hat{a},\hat{b}\right),F_{BTD}\left(x_i,\hat{\alpha},\hat{\theta},\hat{\beta},\hat{\lambda},\hat{a},\hat{b}\right)-\frac{i-1}{n}\right].$$

Table 3: Descriptive Statistics for the data set.

n	Mean	Median	SD	Vari.	Ske.	Kurt.	Min.	Max.
128	9.366	6.395	10.508	110.425	3.287	18.483	0.08	79.05

The statistics $A^*$and W^* are described in details in Chen & Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data. Table 4 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters. Table 5 lists the goodness-of-fits statistics from the fitted models.

Table 4: MLEs and their standard errors (in parentheses) for the data set.

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
BTD	1.27124	0.19211	1.22086	0.60101	0.89641	0.79715
	(0.01176)	(0.33037)	(0.64109)	(0.42042)	(0.04386)	(0.03202)
BD	0.94883	1.34553	0.05548	-	0.65840	0.57668
	(0.08572)	(0.47009)	(0.79009)	-	(0.07601)	(0.06433)
TD	1.16397	2.70655	0.97023	0.58200	-	-
	(0.10300)	(0.64010)	(0.95317)	(0.46134)	-	-
D	1.08554	0.19354	1.10559	-	-	-
	(0.15830)	(0.71774)	(1.12413)	-	-	-

Table 5 shows that the BTD distribution could be chosen as the best model among the fitted models since these models have the lowest values of the -2ℓ**,** AIC, CAIC, BIC, HQIQ, A^*, W^* and KS.

Table 5: The statistics AIC, CAIC, BIC, HQIC, A^*, W^* and K-S for the data set.

Distribution	-2ℓ	AIC	CAIC	BIC	HQIQ	A^*	W^*	K-S
BTD	959.26	969.26	969.75	983.52	975.05	0.3558	0.057	0.026
BD	975.71	987.71	988.39	1004.82	994.66	0.5068	0.074	0.057
TD	1015.06	1007.06	1006.74	995.651	1002.4	0.8444	0.094	0.085
D	1076.76	1082.76	1082.96	1091.32	1086.2	1.3296	0.154	0.138

7. Concluding Remarks

In this study, we have introduced the so-called beta transmuted Dagum (BTD) distribution. This is a generalization of the transmuted Dagum distribution using the genesis of the beta distribution. Many distributions including Dagum, beta Dagum, and transmuted Dagum are embedded in this newly developed BTD distribution. We provide some of its mathematical properties including the moments, moment generating functions, quantile function and Stress-strength model have been provided. We discuss the maximum likelihood estimation of the model parameters. The maximum likelihood estimation procedure is presented. We assess the performance of the maximum likelihood estimators in terms of biases and mean square of errors by means of simulation studies. The usefulness of the new models is illustrated by means of three real data sets. The new model provide consistently better fits than other competitive models for these data sets.

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