ON SOME PROPERTIES OF EXPONENTIATED GENERALIZED GOMPERTZ-MAKEHAM DISTRIBUTION*

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Abstract

In this paper an exponentiated generalised Gompertz-Makeham distribution. An exponentiated generalised family was introduced by Codeiro, et. al., which allows greater flexibility in analysis of data. Some Mathematical and Statistical properties including cumulative distribution function, hazard function and survival function of the distribution are derived. The estimation of model parameters are derived via maximum likelihood estimate method.

Keywords: exponentiated generalised Gompertz-Makeham (EX-GGM) distribution, exponentiated class, probability density function, survival function, hazard function.

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1 Introduction

Recent developments known as generalised class of distributions aid construction of new class of distribution by adding one or more shape parameters to the baseline distribution which makes the new distribution more flexible at its tail region and these enable the new distribution to capture data sets that the existing distribution may not be able to capture. One of the new generated class of distributions is the exponentiated generalised family of distribution.This study focuses on the exponentiated generalised Gompertz-Makeham distribution, which is derived by raising the cumulative distribution function (cdf) of an arbitrary parent distribution to an additional parameter say α. Exponentiated class of distributions are also generalised distributions. The use of exponentiated model started in the early $90s$. Many authors used the exponentiated class of distributions to derive a new class of distributions. Mudholkar & Srivastava (1993) and also Ahuja & Nash (1967) used the exponentiated class of distribution to derive exponentiated weibull distribution; Gupta & Kundu (2001) generalised the Exponential distribution to obtain exponentiated exponential; Anake *et al.* (2015); Lemonte & Cordeiro (2011) and Benkhelifa (2017) presented the exponentiated generalized inverse gaussian, fractional beta-exponential and Marshall-Olkin extended generalized Gompertz-Makeham distribution respectively. The cdf of the exponentiated distribution is given accordingly as:

$$F(x) = (G(x))^{\alpha} \tag{1}$$

and by differentiating equation (1) the pdf of the exponentiated distribution is obtained as

$$f(x) = \alpha g(x)(G(x))^{\alpha-1}, \quad \alpha > 0 \tag{2}$$

If X is a random variable from the normal distribution then for $\alpha = 2$ we have a case of skew normal distribution.

The exponentiated generalised is an extension of the exponentiated class of distributions El-Gohary *et al.* (2013); da Silva *et al.* (2015); Abu-Zinadah & Aloufi (2014); Jafari *et al.* (2014) and Nadarajah & Kotz (2006). The cumulative density function is defined by Cordeiro *et al.* (2013) as:

$$F(x) = [1 - [1 - G(x)]^{\alpha}]^{\beta} \tag{3}$$

and the pdf as,when the cdf is differentiated

$$f(x) = \alpha\beta g(x)[1 - G(x)]^{\alpha-1}[1 - (1 - G(x))^{\alpha}]^{\beta-1}, \quad \alpha > 0, \quad \beta > 0. \tag{4}$$

if $\alpha = 1$ the exponentiated generalised becomes the exponentiated class of distribution. According to Cordeiro *et al.* (2013) the class of exponentiated generalised shares an attractive physical interpretation whenever α and β are positive integers. Considering a device made of independent components in a parallel system with each component made of independent sub-components identically distributed according to $G(x)$ in a series system. The device fails if all components fail and each component fails if any sub-component fails. Let $X_{j1}..., Xj\alpha$ denote the lifetimes of the sub-components within the j^{th} component, $j = 1, ..., \beta$ with common cdf $G(x)$. Let X_j denote the lifetime of the j^{th} component and let X denote the lifetime of the device. Therefore, the cdf of X is

$$P(X \geq x) = P(X_1 \leq x, ..., X_{\beta} \leq x) = P(X_1 \leq x)^{\beta}$$
$$P(X \geq x) = [1 - P(X_1 > x)]^{\beta} = [1 - P(X_{11} > x, ...X1\alpha > x)]^{\beta}$$

$$P(X \geq x) = [1 - P(X_{11} > x)^{\alpha}]^{\beta} = [1 - (1 - P(X_{11} > x)^{\alpha}]^{\beta} \tag{5}$$

The lifetime system of the device obeys the exponentiated generalised family.
The Gompertz-Makeham distribution is a combination of Gompertz and Makeham functions.

Definition 1.1 *A random variable X is Gompertz-Makeham Distributed, If its cumulative Distribution function is given as;*

$$F(x) = 1 - e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)}, \quad x > 0, \quad \lambda > 0, \quad \sigma > 0, \quad \kappa > 0. \tag{6}$$

The Gompertz-Makeham Distribution Probability Density Function is obtained by differentiating equation (6)*;*

$$f(x) = (\sigma e^{\kappa x} + \lambda) e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)}, \tag{7}$$

2 The Exponentiated Generalised Gompertz-Makeham Distribution

Definition 2.1 *The random variable X is said to have exponentiated generalized Gompertz-Makeham Distribution (Ex-GGM) given:*

$$F(x) = [1 - [1 - G(x)]^{\alpha}]^{\beta}, \qquad \alpha, \beta > x \geq 0 \tag{8}$$

and also the corresponding pdf is given as

$$f(x) = \alpha\beta g(x)[1 - G(x)]^{\alpha - 1}[1 - (1 - G(x))^{\alpha}]^{\beta - 1} \tag{9}$$

so, the cdf of the corresponding proposed exponentiated generalized Gompertz-Makeham is

$$F(x) = [1 - [1 - (1 - e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)})]^{\alpha}]^{\beta}$$

for $x, \alpha, \beta, \lambda, \sigma, \kappa > 0$ *then*

$$\Rightarrow F(x) = [1 - [e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)}]^{\alpha}]^{\beta}. \tag{10}$$

The corresponding exponentiated generalised Gompertz-Makeham distribution pdf is derived by substituting the $g(x)$ *and* $G(x)$ *of the Gompertz-Makeham distribution*

$$f(x, \alpha, \beta, \lambda, \sigma, \kappa) = \alpha\beta g(x)[1 - G(x)]^{\alpha - 1}[1 - (1 - G(x))^{\alpha}]^{\beta - 1}$$

$$\begin{aligned} f(x, \alpha, \beta, \lambda, \sigma, \kappa) = \alpha\beta(\sigma e^{\kappa x} + \lambda) e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)} [e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)}]^{\alpha - 1} \\ \times [1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)})^{\alpha}]^{\beta - 1} \end{aligned} \tag{11}$$

$$\Rightarrow f(x, \alpha, \beta, \lambda, \sigma, \kappa) = \alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)}]^{\alpha}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)})^{\alpha}]^{\beta - 1} \tag{12}$$

2.1 Expansion of CDF and PDF of Ex-GGMD in Series

Following the work of Cordeiro *et al.* (2013), where he defined the exponentiated generalised family expansion cdf and pdf as;

$$F(x) = \sum_{j=0}^{\infty} w_j G(x)^j \tag{13}$$

where the coefficients

$$w_j(\alpha, \beta) = \sum_{k=0}^{\infty} \frac{(-1)^{j+k}\Gamma(\beta+1)\Gamma(\alpha j+1)}{\Gamma(\beta-k+1)\Gamma((\alpha-1)j+1)k!j!} \tag{14}$$

and the pdf series expansion

$$f(x) = \alpha\beta g(x) \sum_{j=0}^{\infty} t_j G(x)^j \tag{15}$$

where the

$$t_j(\alpha, \beta) = \frac{(-1)^j\Gamma(\beta)}{j!} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma((k+1)\alpha)}{\Gamma(\beta-k)\Gamma((k+1)\alpha-j)k!} \tag{16}$$

$$f(x) = \sum_{j=0}^{\infty} t_j^* h_{j+1}(x) \tag{17}$$

where $t_j^* = \alpha\beta \dfrac{t_j}{j+1}$ and $h_{j+1}(x) = (j+1)g(x)G(x)^j$
The general binomial theorem of any non-integer β , power series expansion given by ;

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i z^i = \sum_{i=0}^{\infty} \frac{(-1)^i\Gamma(\beta)}{\Gamma(\beta-i)i!} z^i \tag{18}$$

$|z| < 1$ using the binomial expansion for a positive real power,

$$e^{-x} = \sum_{j=1}^{\infty} (-1)^k \frac{x^k}{k!} \tag{19}$$

the expansion of the cdf and pdf Ex-GGM by Applying (18) in (11) then expansion of Ex-GGM cdf becomes

$$F(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha j} \tag{20}$$

$$F(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{j+k} \binom{\beta}{j} \frac{(\alpha j)^k(\lambda x + \frac{\sigma}{\kappa}(e^{\kappa x}-1))^k}{k!} \tag{21}$$

and applying same method to Ex-GGM pdf, it becomes

$$f(x) = \alpha\beta \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(\beta)}{\Gamma(\beta-j)j!} (\sigma e^{\kappa x} + \lambda)(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha(j+1)} \tag{22}$$

$$f(x) = \alpha\beta \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{j+k} \frac{\Gamma(\beta)}{\Gamma(\beta-j)j!} \frac{\alpha(j+1)}{j!k!} (\sigma e^{\kappa x} + \lambda)(\lambda x + \frac{\sigma}{\kappa}(e^{\kappa x}-1))^{\alpha(j+1)} \tag{23}$$

The plot for the pdf and cdf for equation 10 and 12 are shown in Figure 1 for various shape parameters α and β. In the figure it was observed that as $\beta > 1$ the shape of the graph form a unimodal shape or an inverted bathtub shape(bell shaped) else it is monotonously decreasing in shape as $0 < \beta \leq 1$. It was also observed that the graph of the exponentiated generalised Gompertz-Makeham is positively skewed and asymmetric in nature with a heavy tail behaviour to the right,It is a heavy tailed distribution, meaning that a random variable following the Ex-GGMD can have extreme values.

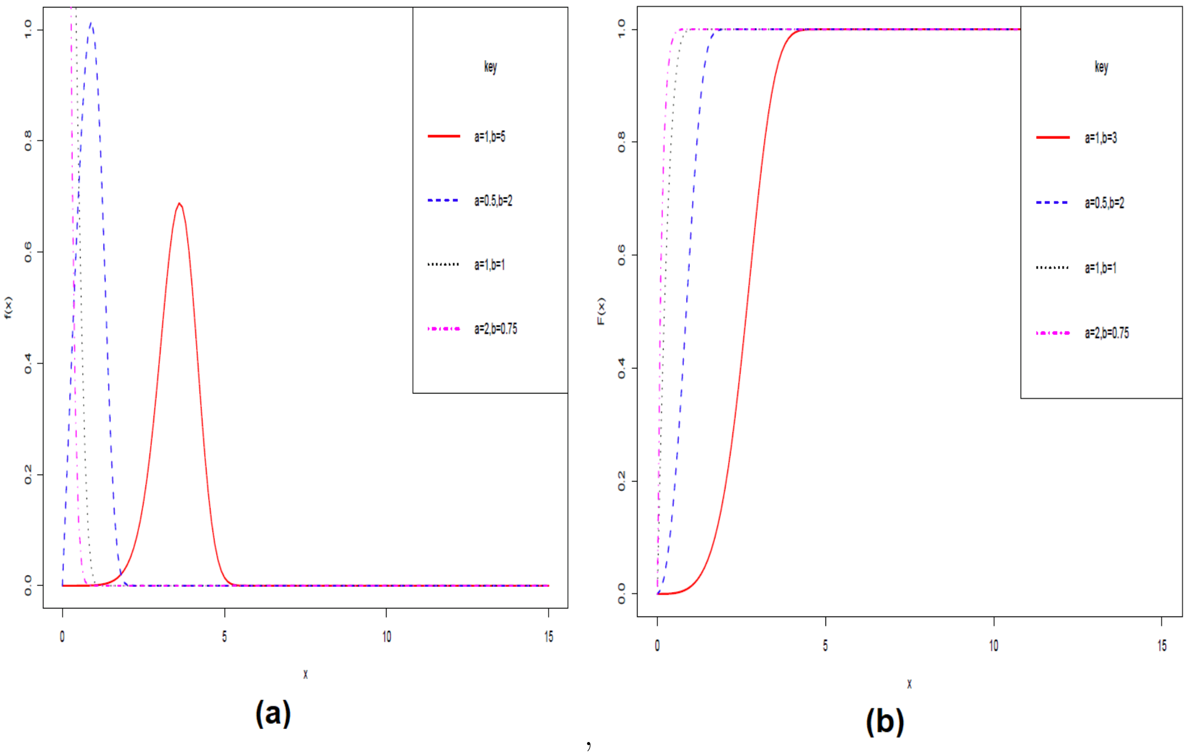

Figure 1: The plot for (a) PDF EX-GGM graph (b) CDF EX-GGM graph.

3 Some Statistical aspect of exponentiated generalized Gompertz-Makeham Distribution

3.1 Quantile Function

The quantile function helps to generate the occurrences of distribution like the median,lower quantile, upper quantile e.t.c. and also helps to obtain the measurement of the skewness and kurtosis

$$F(x) = [1 - [e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}]^{\beta}$$

Let $F(x) = U$.The above equation can not be solved analytically but can be solved numerically using Maple program

$$x = \frac{\sigma}{\kappa\lambda} - \frac{1}{\lambda} ln(1 - U^{\frac{1}{\beta}})^{\frac{1}{\alpha}} - \frac{1}{\kappa} W_0[\frac{\sigma e^{\frac{\sigma}{\kappa}} (1 - U^{\frac{1}{\beta}})^{\frac{-\kappa}{\alpha\lambda}}}{\lambda}] \tag{24}$$

W_0 is lambert function.
The analysis of the variability of the skewness and kurtosis on the shape parameters α and β can be investigated based on quantile measures. Bowley skewness based on quantiles is given by $B = \dfrac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$
The Moors kurtosis based on octile is given by
$M = \dfrac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}$

3.2 Mode

To obtain the mode of exponentiated generalised Gompertz-Makeham distribution, take the logarithms of the pdf of the distribution. i.e.

$$\begin{aligned} log f(x) = n\log\alpha + n\log\beta + \log(\sigma e^{\kappa x} + \lambda) + \alpha(-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)) \\ +(\beta - 1)\log[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}] \end{aligned} \tag{25}$$

differentiating the $\log f(x)$ with respect to x then

$$\frac{\sigma\kappa e^{\kappa x}}{(\lambda + \sigma e^{\kappa x})} - \alpha(\lambda + \sigma e^{\kappa x}) + \frac{\alpha(\beta - 1)(\lambda + \sigma e^{\kappa x})(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}}{1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}} = 0$$

3.3 Moments

Let X be a random variable having Ex-GGMD. Then the rth mean is defined as

$$U^r = E(x^r) = \textstyle\int_y^{\infty} x^r f(x)dx, \quad y \geq 0$$

$$\int_y^{\infty} x^r f(x)\delta x = \alpha\beta \int_y^{\infty} x^r(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1}dx \tag{26}$$

$$\int_y^{\infty} x^r f(x)dx = \sum_{j=0}^{\infty}(-1)^j \binom{\beta - 1}{j} \alpha\beta \int_y^{\infty} x^r(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}dx \tag{27}$$

To solve the integral part

$$\begin{aligned} \int_y^{\infty} x^r(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}dx = \int_y^{\infty} x^r \sigma e^{\kappa x}[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}dx \\ + \int_y^{\infty} x^r \lambda[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}dx \end{aligned} \tag{28}$$

Then we have;

$\int_y^{\infty} x^r(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}dx =$
$\dfrac{r}{\kappa^r} \sum_{p=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}(\dfrac{\kappa}{\sigma})^{i+p} \begin{pmatrix} (\dfrac{-\lambda}{\kappa})(\alpha(j+1)) \\ p \end{pmatrix} \int_{\frac{\sigma}{\kappa}(e^{\kappa y}-1)}^{\infty} z^{i+p}e^{-z\alpha(j+1)}\delta z +$
$\dfrac{\lambda r}{\sigma\kappa^r} \sum_{p=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}(\dfrac{\kappa}{\sigma})^{i+p} \begin{pmatrix} (\dfrac{-\lambda}{\kappa} - 1)(\alpha(j+1)) \\ p \end{pmatrix} \int_{\frac{\sigma}{\kappa}(e^{\kappa y}-1)}^{\infty} z^{i+p}e^{-z\alpha(j+1)}\delta z$

$\Gamma(n) = x^{n-1}e^{-x}\delta x$
Note;

$$\int_{\frac{\sigma}{\kappa}(e^{\kappa y}-1)}^{\infty} z^{i+p}e^{-z\alpha(j+1)}\delta z = \frac{1}{\alpha(j+1)}\Gamma(i + p + 1, \frac{\sigma}{\kappa}(e^{\kappa y} - 1)) \tag{29}$$

3.4 Mean

The mean is defined as $E(x) = \int x \ f(x)dx$
when r=1 in equation 28 the mean is obtained as,

$$\int_y^{\infty} xf(x)dx = \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} \alpha\beta\frac{1}{\kappa}\sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}(\frac{\kappa}{\sigma})^{i+p}\frac{1}{\alpha(j+1)}\Gamma(i+p+1, \frac{\sigma}{\kappa}(e^{\kappa y}-1))$$
$$\{\binom{(\frac{-\lambda}{\kappa})(\alpha(j+1))}{p} + \frac{\lambda}{\sigma}\binom{(\frac{-\lambda}{\kappa}-1)(\alpha(j+1))}{p}\} \quad (30)$$

3.5 Variance

The variance of a distribution is defined as $E(\sum X^2 - \sum(X)^2)$
when $r = 2$ in equation 28 then

$$\int_y^{\infty} x^2f(x)dx = \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} \alpha\beta\frac{2}{\kappa^2}\sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}(\frac{\kappa}{\sigma})^{i+p}\frac{1}{\alpha(j+1)}\Gamma(i+p+1, \frac{\sigma}{\kappa}(e^{\kappa y}-1))$$
$$\{\binom{(\frac{-\lambda}{\kappa})(\alpha(j+1))}{p} + \frac{\lambda}{\sigma}\binom{(\frac{-\lambda}{\kappa}-1)(\alpha(j+1))}{p}\} \quad (31)$$

and E(x) is the mean obtained in equation 30

3.6 Moment generating function

$E(e^{\theta x}) = \int_a^b e^{\theta x} f(x)\delta x$

$\int_a^b e^{\theta x} f(x)\delta x = \alpha\beta \int_a^b e^{\theta x}(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1}\delta x$

$$\int_a^b e^{\theta x} f(x)\delta x = \alpha\beta\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\int_a^b e^{\theta x}(\sigma e^{\kappa x}+\lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}\delta x \quad (32)$$

to solve the integral part
Teimouri & Gupta (2012);

$\int_a^b e^{\theta x}(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}\delta x =$
$\sigma \int_a^b e^{\theta x} e^{\kappa x}[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}\delta x + \lambda \int_a^b e^{\theta x}[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}\delta x$

$$\Gamma(n) = x^{n-1}e^{-x}\delta x$$

$$\frac{1}{\alpha(j+1)}e^{\frac{\alpha\sigma}{\kappa}(j+1)}\Gamma(\frac{\theta-\alpha\lambda(j+1)+\kappa}{\kappa}, \frac{\sigma}{\kappa}(e^{\kappa b}-e^{\kappa a}) + \frac{\lambda}{\alpha(j+1)\sigma}e^{\frac{\alpha\sigma}{\kappa}(j+1)}$$
$$\Gamma(\frac{\theta-\alpha\lambda(j+1)}{\kappa}, \frac{\sigma}{\kappa}(e^{\kappa b}-e^{\kappa a}) \quad (33)$$

substitute equation 33 into 32,then you have the moment generating function of Ex-GGM

3.7 The Characteristic Function

The characteristic function is defined as

$$\phi_x t = E(e^{itx}) = \int_a^b e^{itx} f(x)\delta x$$

$$\int_a^b e^{itx} f(x)\delta x = \alpha\beta \int_a^b e^{itx}(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1-(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1}\delta x$$

$$\int_a^b e^{itx} f(x)\delta x = \alpha\beta \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} \int_a^b e^{itx}(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(j+1)}\delta x \tag{34}$$

$$\frac{1}{\alpha(j+1)} e^{\frac{\alpha\sigma}{\kappa}(j+1)}\Gamma(\frac{it-\alpha\lambda(j+1)+\kappa}{\kappa}, \frac{\sigma}{\kappa}(e^{\kappa b} - e^{\kappa a}) + \frac{\lambda}{\alpha(j+1)\sigma} e^{\frac{\alpha\sigma}{\kappa}(j+1)} \Gamma(\frac{it-\alpha\lambda(j+1)}{\kappa}, \frac{\sigma}{\kappa}(e^{\kappa b} - e^{\kappa a}) \tag{35}$$

substitute equation 35 into 34,then you have the Characteristic function of Ex-GGM

3.8 The cumulant generating function

it is defined by taking the Natural logarithm of the Moment generating function as; $C_x\theta = logE(e^{\theta x})$

$$C_x\theta = log\alpha + log\beta + log(\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} (\frac{1}{\alpha(j+1)} e^{\frac{\alpha\sigma}{\kappa}(j+1)} \Gamma(\frac{\theta-\alpha\lambda(j+1)+\kappa}{\kappa}, \frac{\sigma}{\kappa}(e^{\kappa b} - e^{\kappa a})) + \frac{\lambda}{\alpha(j+1)\sigma} e^{\frac{\alpha\sigma}{\kappa}(j+1)}\Gamma(\frac{\theta-\alpha\lambda(j+1)}{\kappa}, \frac{\sigma}{\kappa}(e^{\kappa b} - e^{\kappa a})))) \tag{36}$$

3.9 Asymptotic Behaviour of Exponentiated Generalised Gompertz-Makeham Distribution

We seek to investigate the behaviour of the proposed model as $x \to 0$ and $x \to \infty$.

$$f(x) =_{\lim_{x\to 0}} \alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1-(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1} = 0 \tag{37}$$

As

$$f(x) =_{\lim_{x\to\infty}} \alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1-(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1} = 0 \tag{38}$$

These results confirm further that the distribution has a mode

As
$\beta = 1$; in equation 38

$$f(x) = \alpha(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha} \tag{39}$$

$$f(x) = \lim_{x \to 0} \alpha(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}$$

$$f(x) = \alpha(\sigma + \lambda) \tag{40}$$

$\alpha = 1$; in equation 40

$$f(x) = (\sigma + \lambda) \tag{41}$$

3.10 Survival function of exponentiated generalized Gompertz-Makeham distribution

The survival function is defined as:

$$S(x) = 1 - F(x) \tag{42}$$

substitute the cdf of EX-GGM into equation 42

$$S(x) = 1 - [1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta}$$

As $\beta = 1$ then

$$S(x) = e^{\alpha(-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1))} \tag{43}$$

$\lim_{x \longrightarrow 0} S(x) = 1$ and $\lim_{x \longrightarrow \infty} S(x) = 0$
the range of survival function is $0 \leq S(x) \leq 1$

3.11 Reversed hazard rate

The reversed hazard rate is defined as the ratio of the probability density function and the corresponding cumulative distribution function.The reversed hazard rate is used for examining the nature of the probability functions and application can be used in finance, forensic science, in actuarial sciences, etc.

$$r(x) = \frac{f(x)}{F(x)} \tag{44}$$

substituting the cdf and pdf of the exponentiated generalised Gompertz-Makeham in equation 11 and 12 into eqaution 44,

$$r(x) = \frac{\alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1}}{[1 - [e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}]^{\beta}} \tag{45}$$

as $\beta = 1$ we have

$$r(x) = \frac{\alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}}{[1 - [e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}]} \tag{46}$$

in equation 46

$\lim_{x \longrightarrow 0} r(x) = \infty$ and $\lim_{x \longrightarrow \infty} r(x) = 0$

3.12 Hazard function of Exponentiated Generalized Gompertz-Makeham function

The hazard function is defined as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. Specifically, it is computed as the number of failures per unit time in the respective interval, divided by the average number of surviving cases at the mid-point of the interval. Mathematically, the hazard function for a random variable X is defined as:

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{47}$$

substituting the pdf and cdf of Gompertz-Makeham Distribution into equation 47 the hazard function becomes

$$h(x) = \frac{\alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1}}{1 - [1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta}} \tag{48}$$

$\beta = 1$ in equation 48

$$h(x) = \frac{\alpha(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}}{[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}} \tag{49}$$

then the hazard function becomes

$$h(x) = \alpha(\sigma e^{\kappa x} + \lambda) \tag{50}$$

The behaviour of the hazard function as x approaches zero and as x approaches ∞ in equation 50 are as follows:

$$h(x) = \lim_{x \longrightarrow 0} \alpha(\sigma e^{\kappa x} + \lambda) = \alpha(\sigma + \lambda) \tag{51}$$

and as

$$h(x) = \lim_{x \longrightarrow \infty} \alpha(\sigma e^{\kappa x} + \lambda) = \infty \tag{52}$$

The plot for Hazard function graph, Reversed hazard rate graph, and Survival function graph presented at Figure 2.

4 Order statistics

Let $X_1, X_2..., X_n$ be a random sample from the exponentiated generalised Gompertz-Makeham distribution with the cdf and pdf in equation 11 and 12 above respectively. Let $X_{1:n} \leq X_{2:n} \leq X_{3:n}... \leq X_{n:n}$ denote the order statistics obtained from the sample then $f_{1:n}(x)$ of $X_{i:n}$ for i=1,2,...,n. The probability density function of $X_{i:n}$ is given by

$$f_{i:n}(x, \alpha, \beta, \sigma, \kappa, \lambda) = \frac{f(x)}{B(i, n-i+1)}[F(x)]^{i-1}[1 - F(x)]^{n-i} \tag{53}$$

F(x) and f(x) are the cdf and pdf of the exponentiated generalised Gompertz-Makeham distribution respectively, and B(.,.) is the beta function. This implies that:

$$f_{i:n}(x) = \frac{\alpha\beta}{B(i, n-i+1)} g(x)[1 - G(x)]^{\alpha-1}[1 - (1 - G(x))^{\alpha}]^{\beta i-1} \\ (1 - (1 - (1 - G(x))^{\alpha})^{\beta})^{n-i} \tag{54}$$

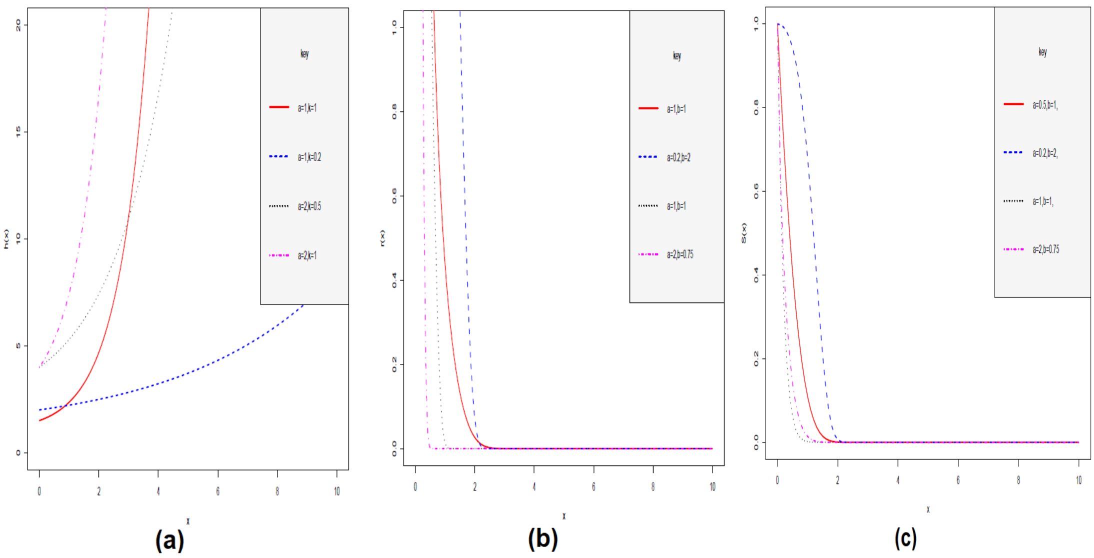

Figure 2: The plot for: (a) Hazard function graph, (b) Reversed hazard rate graph, and (c) Survival function graph.

Using the binomial series expansion for the last term in equation 54

$$(1-z)^{\beta-1} = \Sigma_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} z^i = \Sigma_{i=0}^{\infty} \frac{(-1)^i\Gamma(\beta)}{\Gamma(\beta-i)i!} z^i$$

$$f_{i:n}(x) = \sum_{p=0}^{n-i}(-1)^p \binom{n-i}{p} \frac{\alpha\beta}{B(i,n-i+1)} g(x)[1-G(x)]^{\alpha-1} \\ [1-(1-G(x))^{\alpha}]^{\beta(i+p)-1} \tag{55}$$

Again applying the binomial expansion to the last term in equation 55

$$f_{i:n}(x) = \sum_{p=0}^{n-i}\sum_{q=0}^{\infty}(-1)^{p+q} \binom{n-i}{p} \binom{\beta(i+p)-1}{q} \frac{\alpha\beta}{B(i,n-i+1)} g(x) \\ [1-G(x)]^{\alpha(q+1)-1} \tag{56}$$

Finally, apply the expansion on the last term equation 56

$$f_{i:n}(x) = \sum_{p=0}^{n-i}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{p+q+r} \binom{n-i}{p} \binom{\beta(i+p)-1}{q} \binom{\alpha(q+1)-1}{r} \\ \frac{\alpha\beta}{B(i,n-i+1)} g(x)G(x)^r \tag{57}$$

$$f_{i:n}(x) = \frac{\alpha\beta}{B(i,n-i+1)} \sum_{r=0}^{\infty} Z_r g(x)G(x)^r \tag{58}$$

where

$$Z_r = \sum_{p=0}^{n-i}\sum_{q=0}^{\infty}(-1)^{p+q+r} \binom{n-i}{p} \binom{\beta(i+p)-1}{q} \binom{\alpha(q+1)-1}{r} \tag{59}$$

$G(x)$ and $g(x)$ in equation 59 are the cdf and pdf of Gompertz-makeham distribution in equation 6 and 7 recpectively.
The pdf of order statistics are observed to be equal to the density of the Gompertz-Makeham distribution multiplied by an infinite weighted power series of cdfs for the Gompertz-Makeham distribution

The rth moment of $E(X_{i:n}^r)$ is given by $E(X_{i:n}^r) = \int_{-\infty}^{\infty} x^r f_{i:n}(x)$
therefore

$$E(X_{i:n}^r) = \frac{\alpha\beta}{B(i, n-i+1)} \sum_{r=0}^{\infty} Z_r^* \int_{-\infty}^{\infty} x^r h_{r+1} \tag{60}$$

and the moment generating function of $E(X_{i:n}^r)$ is given by

$$M(t) = \frac{\alpha\beta}{B(i, n-i+1)} \sum_{r=0}^{\infty} Z_r^* \int_{-\infty}^{\infty} e^{tx} h_{r+1} \tag{61}$$

where $Z_r^* = \dfrac{Z_r}{r+1}$ and $h_{r+1} = (r+1)g(x)G(x)^r$

5 Entropies

An entropy is a measure of variation or uncertainty of a random variable. There are two well known entropies, Renyi and Shannon entropy.
The Shannon entropy is defined as $E[-logf(x)] = -\int_{-\infty}^{\infty} f(x)logf(x)dx$
so if X is a random variable, the Shannon entropy exponentiated generalised Gompertz-Makeham distribution is given as

$$\begin{aligned} E[-log(f(x))] = -log(\alpha\beta) - E(\sum_{i=0}^{\infty} log(\lambda + \sigma e^{\kappa x}) - \alpha \sum_{i=0}^{\infty}(-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1)) \\ -(\beta - 1)log(\sum_{i=0}^{\infty}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}])) \end{aligned} \tag{62}$$

The Renyi Entropy is defined as

$$I_R(c) = \frac{1}{(1-c)} log(\int_{-\infty}^{\infty} f(x)^c \delta x) \tag{63}$$

$$f(x)^c = (\alpha\beta)^c(\sigma e^{\kappa x} + \lambda)^c[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{c\alpha}[1 - [e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}]^{c(\beta-1)}$$

$$f(x)^c = (\alpha\beta)^c \textstyle\sum_{i=0}^{\infty}(-1)^i \binom{c(\beta-1)}{i} (\sigma e^{\kappa x} + \lambda)^c[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha(c+i)}$$

$$f(x)^c = (\alpha\beta)^c \sum_{i=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{i+r} \binom{c(\beta-1)}{i} \frac{(\frac{\alpha\sigma}{\kappa}(c+i))^r}{r!} (\sigma e^{\kappa x} + \lambda)^c[e^{-\lambda x - \frac{\sigma}{\kappa}e^{\kappa x}}]^{\alpha(c+i)} \tag{64}$$

substitute equation 64 in equation 63,then becomes

$$I_R(c) = \frac{1}{(1-c)} log(\alpha\beta)^c \sum_{i=0}^{\infty} t_{i,r} \int_{-\infty}^{\infty} (\sigma e^{\kappa x} + \lambda)^c[e^{-\lambda x - \frac{\sigma}{\kappa}e^{\kappa x}}]^{\alpha(c+i)} \delta x \tag{65}$$

$$t_{i,j} = \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{i+r} \binom{c(\beta-1)}{i} \frac{(\frac{\alpha\sigma}{\kappa}(c+i))^r}{r!}$$

$$\int_{-\infty}^{\infty} (\sigma e^{\kappa x} + \lambda)^c [e^{-\lambda x - \frac{\sigma}{\kappa} e^{\kappa x}}]^{\alpha(c+i)} \delta x = \binom{c}{i} \frac{\lambda^{c-1}\sigma^i}{\kappa} \Gamma((i - \frac{\lambda}{\kappa})(\alpha(c+i)), (\alpha\frac{\sigma}{\kappa}(c+i))) \quad (66)$$

substitute equation 66 into equation 65 the Renyi entropy becomes

$$I_R(c) = \frac{1}{(1-c)} log((\alpha\beta)^c \sum_{i=0}^{\infty} t_{i,r} (\binom{c}{i} \frac{\lambda^{c-1}\sigma^i}{\kappa} \Gamma((i - \frac{\lambda}{\kappa})(\alpha(c+i)), (\alpha\frac{\sigma}{\kappa}(c+i)))) \quad (67)$$

6 Inferential aspect of the model

6.1 Parameter estimation using maximum likelihood estimate method

The parameters of the exponentiated generalized Gompertz-Makeham distribution can be estimated using the method of maximum likelihood estimation (MLE)
Let $x_1, x_2, ...x_n$ denote a random sample of size n from the exponentiated generalised Gompertz-Makeham distribution.
The likelihood function is derived by taking the logarithm of the probability density function of the exponentiated generalised Gompertz-Makeham function and also take the derivative of each parameters;

$$L(\hat{X}|\alpha, \beta, \sigma, \kappa, \lambda) = \prod_{i=1}^{n} [\alpha\beta(\sigma e^{\kappa x} + \lambda)[e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)}]^{\alpha}[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}]^{\beta-1}] \quad (68)$$

$$\log L(\hat{X}|\alpha, \beta, \sigma, \kappa, \lambda) = n \log\alpha + n\log\beta + \sum_{i=1}^{n} \log(\sigma e^{\kappa x} + \lambda)$$
$$+ \alpha \sum_{i=1}^{n} \log e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)} + \beta - 1 \sum_{i=1}^{n} \log[1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}] \quad (69)$$

differentiating the likelihood function to each of the parameters : $\alpha, \beta, \sigma, \kappa, \lambda$ gives;
if $l = \log L(\hat{X}|\alpha, \beta, \sigma, \kappa, \lambda)$ then

$$\frac{\delta l}{\delta\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} (-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x} - 1))(1 - \frac{(\beta-1)(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}}{1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}}) \quad (70)$$

$$\frac{\delta l}{\delta\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log\{1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}\} \quad (71)$$

$$\frac{\delta l}{\delta\sigma} = \sum_{i=1}^{n} [\frac{e^{\kappa x} - \frac{\alpha}{\kappa}(e^{\kappa x} - 1)(\sigma e^{\kappa x} + \lambda)}{(\sigma e^{\kappa x} + \lambda)} - \frac{\alpha(\beta-1)(e^{\kappa x} - 1)(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}}{\kappa(1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha})}] \quad (72)$$

$$\frac{\delta l}{\delta\lambda} = \sum_{i=1}^{n} [\frac{1 - \alpha x(\sigma e^{\kappa x} + \lambda)}{(\sigma e^{\kappa x} + \lambda)} + \frac{\alpha x(\beta-1)(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}}{(1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha})}] \quad (73)$$

$$\frac{\delta l}{\delta\kappa} = \sum_{i=1}^{n} [\frac{\sigma x e^{\kappa x}}{(\sigma e^{\kappa x} + \lambda)} - \frac{\alpha(\beta-1)(\sigma(e^{\kappa x} - 1) - \sigma k x e^{\kappa x})(e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha}}{\kappa^2(1 - (e^{-\lambda x - \frac{\sigma}{\kappa}(e^{\kappa x}-1)})^{\alpha})}] \quad (74)$$

the solutions of $\frac{\delta l}{\delta\alpha} = 0, \frac{\delta l}{\delta\beta} = 0, \frac{\delta l}{\delta\sigma} = 0$, $\frac{\delta l}{\delta\kappa} = 0, \frac{\delta l}{\delta\lambda} = 0$ gives the maximum likelihood estimate for parameters $a, b, \sigma, \kappa, \lambda$ respectively. Since there is no explicit solutions to the equation, we resolved into into numerical package using Maple
For interval estimation and hypothesis tests on model parameters, we require the Fisher's information matrix, the 5 by 5 unit observed information matrix $I = J_n(.,.)$
For estimation and hypothesis tests on model parameters, we require the observed information matrix, the 5 by 5 unit observed information matrix $I = J_n(.,.)$

$$I = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\sigma} & J_{\alpha\lambda} & J_{\alpha\kappa} \\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\sigma} & J_{\beta\lambda} & J_{\beta\kappa} \\ J_{\sigma\alpha} & J_{\sigma\beta} & J_{\sigma\sigma} & J_{\sigma\lambda} & J_{\sigma\kappa} \\ J_{\lambda\alpha} & J_{\lambda\beta} & J_{\lambda\sigma} & J_{\lambda\lambda} & J_{\lambda\kappa} \\ J_{\kappa\alpha} & J_{\kappa\beta} & J_{\kappa\sigma} & J_{\kappa\lambda} & J_{\kappa\kappa} \end{pmatrix} \tag{75}$$

$$I = \begin{pmatrix} var(\hat{\alpha\alpha}) & cov(\hat{\alpha\beta}) & cov(\hat{\alpha\sigma}) & cov(\hat{\alpha\lambda}) & cov(\hat{\alpha\kappa}) \\ cov(\hat{\beta\alpha}) & var(\hat{\beta\beta}) & cov(\hat{\beta\sigma}) & cov(\hat{\beta\lambda}) & cov(\hat{\beta\kappa}) \\ cov(\hat{\sigma\alpha}) & cov(\hat{\sigma\beta}) & var(\hat{\sigma\sigma}) & cov(\hat{\sigma\lambda}) & cov(\hat{\sigma\kappa}) \\ cov(\hat{\lambda\alpha}) & cov(\hat{\lambda\beta}) & cov(\hat{\lambda\sigma}) & var(\hat{\lambda\lambda}) & cov(\hat{\lambda\kappa}) \\ cov(\hat{\kappa\alpha}) & cov(\hat{\kappa\beta}) & cov(\hat{\kappa\sigma}) & cov(\hat{\kappa\lambda}) & var(\hat{\kappa\kappa}) \end{pmatrix} \tag{76}$$

The information matrix were obtained by taking the second derivative in respect to their parameters. Therefore an $100(1-q)$ asymptotic intervals for parameters$\alpha, \beta, \lambda, \kappa, \sigma$ $\alpha \pm Z_{\frac{q}{2}}\sqrt{var(\hat{\alpha})}$, $\beta \pm Z_{\frac{q}{2}}\sqrt{var(\hat{\beta})}$, $\lambda \pm Z_{\frac{q}{2}}\sqrt{var(\hat{\lambda})}$, $\kappa \pm Z_{\frac{q}{2}}\sqrt{var(\hat{\kappa})}$, $\sigma \pm Z_{\frac{q}{2}}\sqrt{var(\hat{\sigma})}$
var(.) is the diagonal of matrix J^{-I} corresponding to each parameter and $Z_{\frac{q}{2}}$ is the quantile.
The co-variances are obtained as follows For internal estimation and hypothesis tests on model parameters, we require the observed information matrix, the 5 by 5 unit observed information matrix $I = J_n(.,.)$

7 Conclusion

In this work, a five-parameter exponentiated generalization of Gompertz-Makeham distribution was derived and their properties such as mean, median, mode and distribution of order statistics were also obtained. The distribution was characterized by relating it to other probability distributions and some areas of application of the exponentiated generalization of Gompertz-Makeham distribution were identified. The model is positively skewed, its shape could be decreasing or unimodal (depending on the values of the parameters)
Further studies are on going to derive more generators which can be use to model many other distributions which can be the best in applications.

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Appendices

$$J_{aa} = -\frac{n}{a^2} + \sum_{i=1}^{n} \frac{\left(-\lambda\, x - \frac{\sigma\,(\mathrm{e}^{\kappa\, x}-1)}{\kappa}\right)^2 \left(-\,(b-1)\left(\mathrm{e}^{-\lambda\, x - \frac{\sigma\,(\mathrm{e}^{\kappa\, x}-1)}{\kappa}}\right)^a\right)}{\left(1-\left(\mathrm{e}^{-\lambda\, x-\frac{\sigma\,(\mathrm{e}^{\kappa\, x}-1)}{\kappa}}\right)^a\right)^2} \tag{77}$$

$$J_{ab} = -\sum_{i=1}^{n} \frac{\left(-\lambda\, x - \frac{\sigma\,(\mathrm{e}^{\kappa\, x}-1)}{\kappa}\right)\left(\mathrm{e}^{-\lambda\, x-\frac{\sigma\,(\mathrm{e}^{\kappa\, x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\, x-\frac{\sigma\,(\mathrm{e}^{\kappa\, x}-1)}{\kappa}}\right)^a\right)} \tag{78}$$

$$J_{a\kappa} = \sum_{i=1}^{n} \left(\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa^2} - \frac{\sigma\,x\mathrm{e}^{\kappa\,x}}{\kappa} \right) \left(1 - \frac{(b-1)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)} \right) + \sum_{i=1}^{n} \left(-\lambda\,x - \frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa} \right) \left(\frac{-a\,(b-1)\left(\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa^2} - \frac{\sigma\,x\mathrm{e}^{\kappa\,x}}{\kappa} \right)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)^2} \right) \tag{79}$$

$$J_{a\lambda} = -\sum_{i=1}^{n} x \left(1 - \frac{(b-1)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)} \right) + \sum_{i=1}^{n} \left(-\lambda\,x - \frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa} \right) \left(\frac{ax\,(b-1)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)^2} \right) \tag{80}$$

$$J_{a\sigma} = -\sum_{i=1}^{n} \frac{(\mathrm{e}^{\kappa\,x}-1)}{\kappa} \left(1 - \frac{(b-1)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x})}{\kappa}}\right)^a\right)} \right) + \sum_{i=1}^{n} \left(-\lambda\,x - \frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa} \right) \left(\frac{\frac{a\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}\,(b-1)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)^2} \right) \tag{81}$$

$$J_{bb} = -\frac{n}{b^2} \tag{82}$$

$$J_{b\sigma} = \sum_{i=1}^{n} \frac{\frac{a\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}\left(\mathrm{e}^{-\lambda x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)} \tag{83}$$

$$J_{b\lambda} = \sum_{i=1}^{n} \frac{x\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a} + \sum_{i=1}^{n} \frac{ax\left(\mathrm{e}^{-\lambda x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\left(-\lambda\,x - \frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}\right)}{\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)^2} \tag{84}$$

$$J_{b\kappa} = -\sum_{i=1}^{n} \frac{a\left(\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa^2} - \frac{\sigma\,x\mathrm{e}^{\kappa\,x}}{\kappa}\right)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{1-\left(\mathrm{e}^{-\lambda x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a} \tag{85}$$

$$J_{\sigma\sigma} = \sum_{i=1}^{n} \left(-\frac{(e^{\kappa\,x})^2}{(\sigma\,e^{\kappa\,x}+\lambda)^2} + \frac{a^2(b-1)(e^{\kappa x}-1)^2\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\kappa^2\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)^2} \right) \tag{86}$$

~

$$J_{\sigma\lambda} = \sum_{i=1}^{n} \left(-\frac{e^{\kappa\,x}}{(\sigma\,e^{\kappa\,x}+\lambda)^2} + \frac{a^2x(b-1)(e^{\kappa x}-1)\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a}{\kappa\left(1-\left(\mathrm{e}^{-\lambda\,x-\frac{\sigma\,(\mathrm{e}^{\kappa\,x}-1)}{\kappa}}\right)^a\right)^2} \right) \tag{87}$$

$$J_{\sigma\kappa} = \sum_{i=1}^{n} \left(\frac{\left(xe^{\kappa x} - \frac{axe^{\kappa x}(\sigma e^{\kappa x} + \lambda)}{\kappa} - \frac{a(e^{\kappa x} - 1)\sigma x e^{\kappa x}}{\kappa} + \frac{a(e^{\kappa x} - 1)(\sigma e^{\kappa x} + \lambda)}{\kappa^2} \right)}{(\sigma e^{\kappa x} + \lambda)} \right)$$
$$- \left(\frac{\left(e^{\kappa x} - \frac{a(e^{\kappa x} - 1)(\sigma e^{\kappa x} + \lambda)}{\kappa} \right) \sigma x e^{\kappa x}}{(\sigma e^{\kappa x} + \lambda)^2} \right) - \left(\frac{a(b-1)\left(\kappa x e^{\kappa x} - (e^{\kappa x} - 1)\right) \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\kappa^2 \left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)} \right)$$
$$- \left(\frac{a^2 (\frac{-\sigma x e^{\kappa x}}{\kappa} + \frac{\sigma(e^{\kappa x} - 1)}{\kappa^2})(b-1)(e^{\kappa x} - 1) \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\kappa \left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)^2} \right) \quad (88)$$

$$J_{\lambda\lambda} = \sum_{i=1}^{n} \left(-\frac{1}{(\sigma e^{\kappa x} + \lambda)^2} - \frac{a^2 x^2 (b-1) \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)^2} \right) \quad (89)$$

$$J_{\lambda\kappa} = \sum_{i=1}^{n} \left(-\frac{1}{(\sigma e^{\kappa x} + \lambda)^2} + \frac{a^2 x (b-1) \left(\frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa^2} - \frac{\sigma x \mathrm{e}^{\kappa x}}{\kappa} \right) \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)^2} \right) \quad (90)$$

$$J_{\kappa\kappa} = \sum_{i=1}^{n} \left(-\frac{\sigma x^2 e^{\kappa x} \left((\sigma e^{\kappa x} + \lambda) - \sigma e^{\kappa x} \right)}{(\sigma e^{\kappa x} + \lambda)^2} \right) + \left(\frac{a(b-1)\sigma x^2 e^{\kappa x} \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\kappa \left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)} \right) +$$
$$\left(\frac{2a(b-1)\left(\sigma(e^{\kappa x} - 1) - \sigma\kappa x e^{\kappa x} \right) \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\kappa^3 \left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)} \right) - \quad (91)$$
$$\left(\frac{a^2 (b-1)\left(\sigma(e^{\kappa x} - 1) - \sigma\kappa x e^{\kappa x} \right) \left(\frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa^2} - \frac{\sigma x \mathrm{e}^{\kappa x}}{\kappa} \right) \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a}{\kappa^2 \left(1 - \left(\mathrm{e}^{-\lambda x - \frac{\sigma(\mathrm{e}^{\kappa x} - 1)}{\kappa}} \right)^a \right)^2} \right)$$